The Historical Growth of Symbolic Meaning in Mathematics

Enlightening Symbols: A Short History of Mathematical Notation and its Hidden Powers

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Why review a book on the history of mathematical symbols in a journal on psychology? The answer lies in the relevance of cultural development to the current state of psychology and its evolution, especially in relation to the social, cognitive and affective development of mathematical thinking and its summative and diagnostic assessment.

Joseph Mazur is an author well known for presenting subtle mathematical ideas in highly readable books. It is therefore doubly pleasurable to have the opportunity to review *Enlightening Symbols* on the history of symbols in mathematics alongside his review of my own book on *How Humans Learn to think Mathematically*. Mazur approaches his task by considering the historical development of symbolism and the subtle ways in which symbolism makes new ways of thinking possible, often taking centuries for new ideas to become widely accepted. My book studies the cognitive development of the individual from child to adult mathematician and reveals an overall framework that also applies to the historical development of mathematics.

There are broad parallels between the two books that have serious implications for the psychology of mathematical thinking. In particular different practices in the teaching and learning of mathematics around the world depend on the shared beliefs of the various communities concerned, including a range of specialisms in mathematics, cognitive science, philosophy, psychology, teaching, curriculum design and politics. These beliefs may support or impede the development of mathematical thinking in ways that may be appropriate for some and entirely inappropriate for others.

Enlightening Symbols tells the stories of the development of symbolism in mathematics from pre-history to the present-day notation familiar in arithmetic, algebra and geometry.

After an introduction revealing that, other than numerals, mathematical symbols as we currently know them did not exist before the sixteenth century, the book is divided into three parts: the development of numerals, algebraic symbolism, and the special power of symbols to support highly sophisticated mathematical thinking.

Each of the twenty chapters in the book is a concise analysis of a specific topic, seeking wherever possible to refer to the earliest possible origins, querying subsequent interpretations and formulating reasonable conclusions based on the author's own broad experience.

The first part of the book (80 pages) takes us through successive chapters on the early origins of numbers (where number notation preceded writing by more than a thousand years), and the development of number symbolism and methods of calculation as different communities developed specific ways of representing and operating with number symbols and came in contact with each other through trade. Some used written symbols such as Babylonian markings on clay or Egyptian hieroglyphics that were amenable to certain forms of calculation while others used symbols such as Roman numerals that were less amenable to calculation, using enactive materials such as an abacus or sand tray to carry out arithmetic operations. Cultural differences caused familiar systems in some countries to be maintained, resisting the introduction of more flexible systems that could take centuries to become established. In particular, initial versions of 'Hindu-Arabic' symbols used 9 digits and an empty place marker for many centuries before making the all-important transition to a ten-digit system using 0, 1, 2, 3, 4, 6, 7, 8, 9 to represent any whole number of any size with corresponding algorithms for calculation. But did these ideas originally arise in India to be transferred to Europe by the Arabs, or did the Indians build on earlier ideas from China? Original materials no longer survive and the question may not have a definitive answer.

In each successive chapter the author considers the evidence with a level of skepticism that analyzes why the changes in notation and operation occurred and why they took so long to be implemented. In the final chapter of the first part, he considers the veracity of the evidence and opinions that have been put forward over the generations and considers how unexpected discoveries of previously unknown materials can drastically change our interpretations of the past.

Part 2 of the book (97 pages) begins from the time when mathematics operated using ordinary language without symbols. Euclid formulated a theorem that has been translated as 'if a straight line is cut at random, the square on the whole is equal to the squares on the segments and twice the rectangle contained by the segments.' It took around two thousand years to develop the modern equation $(a + b)^2 = a^2 + b^2 + 2ab$.

Mazur takes us succinctly through successive developments over the centuries as statements are written in more concise yet still cumbersome forms, until Descartes in the seventeenth century makes the link between geometry and algebra by representing points in the plane as pairs of coordinates and using letters of the alphabet to represent constant and variable quantities. From this breakthrough, the introduction of symbols proliferated and new ways of thinking developed to solve previously intractable problems, culminating in the calculus of Newton and Leibniz.

In this review I will not attempt to relate the twists and turns that Mazur narrates with great relish. The underlying story describes how new ways of operating with symbols gives new results that evolve mathematical thinking while raising new questions that are met with suspicion. This evolution continues today as new challenges require new ways of thinking that are ripe for new interpretations in both mathematics and psychology.

The final part of the book (38 pages) reflects on the hidden power of symbols, quoting A. N. Whitehead's insight, 'by relieving the brain of all unnecessary work, a good notation sets it free to concentrate on more advanced problems.' Symbolism does more than that: symbols carry with them a variety of meanings, some of which are supportive in dealing with new challenges but others act as impediments to new thought. Mazur reflects on how the idea of a square root of the number –1 was impossible to conceive before the Italians in the sixteenth century sought to solve quadratic and cubic equations. Then they found that if they worked with a symbol representing the square root of –1, they could find 'imaginary solutions' of the form $a+b\sqrt{-1}$ where a and b are 'real numbers'. This is psychologically very disturbing because non-zero real numbers always have a positive square, so $\sqrt{-1}$ is an unthinkable absurdity. At a later stage, by replacing the symbol $\sqrt{-1}$ by i and manipulating the letter as if it were an algebraic entity, it became possible for mathematicians to manipulate symbols and replace i^2 by –1 to solve new problems with increasing confidence. However, this activity was often accompanied by a sneaky feeling that it gave the right answer for the wrong reason and this underlying distrust continued even when complex numbers could be visualized as points in the plane with i being the point (0,1). Slowly over time, the coherence of complex analysis allowed mathematics to evolve in sophistication.

This brings us to the current state of the evolution of mathematics and the role of psychology in moving into the future. Throughout the whole of the book *Enlightening Symbols*, we see different communities of practice in mathematics progressing over the generations and, at any given time, the experts in different communities will have their own ways of analyzing and explaining the nature of mathematics in science and society. Sometimes experts have views that impede long-term evolution. The same phenomenon applies to us today.

Mathematical symbols as we know them have been with us for only a short period of recent history. Now we have entered an information age where we can use our fingers on a tablet to manipulate data enactively, to represent it in words and dynamic pictures, and to use the inner power of symbols to concentrate on more advanced problems. Our tools are evolving in ways that allow new possibilities that are as yet beyond our collective human capacity to clearly predict the future.

At this point, we can reflect on how the two books *Enlightening Symbols* and *How Humans Learn to Think Mathematically* relate to the current directions of interest in psychology in general and in mathematics in particular.

Psychological studies involve a range of communities of practice, each with its focus of interest, some which differ dramatically and some which may be blended together to give a broader long-term picture. These include earlier forms of gestalt psychology based on the wholeness of structures, behaviorist psychology that studies how animals in general and people in particular can be trained to carry out routine procedures, and more recent forms of cognitive and social constructivism that consider individual and social development. Meanwhile new technology and the increasing capacity for data handling has enabled huge data sets to be collected to compare national progress in an increasingly competitive global economy.

The 1960s saw the arrival of the 'new mathematics' based on the premise that if only we could get the structure of the mathematics right, then learning could be improved. This focus on the structure of mathematics did not have the desired effect, in part because different individuals interpret mathematics in different ways, leading to a constructivist approach in the 1980s in which the child was encouraged to make sense of the mathematics in his or her own way, both competitively and in cooperation with others. This in turn led to the 'Math Wars' in which proponents of traditional mathematics teaching criticized the loss of required mathematical techniques. In the new millennium, the increasing availability of comparative international statistics (TIMSS and PISA) has led to government policies around the world to 'raise standards' to improve international competitiveness in the global economy.

Assessment requires a framework for the specification and evaluation of progress. In the fifties, Bloom and his colleagues began to set out a taxonomy of educational objectives to cover the cognitive, affective and psycho-motor domains. The cognitive domain specified six aspects in increasing sophistication: knowledge, comprehension, application, analysis, evaluation and synthesis. The affective domain focused on five aspects: receiving, responding, valuing, organizing and characterizing. It is interesting to note that this domain covers only positive aspects of attitudes and beliefs and does not mention negative affective aspects, such as mathematics anxiety.

Yet the history of *Enlightening Symbols* shows the successive resistance of various societies to new ideas in mathematics that do not fit easily into their current way of thinking. For example, each stage of the emergence of new number concepts is met with a cultural resistance, such as the development of symbols to represent and operate with whole numbers, the introduction of fractions, decimals, signed numbers, real numbers, the transition from arithmetic to algebra and the later developments of complex numbers and the calculus.

The historical development of mathematics evolves differently from the cognitive development of the individual. It concerns the evolution of ideas shared by adults in successive generations, as developed by gifted mathematicians, while the cognitive development covers the full range of the population as individuals develop in different ways from child to adult. Nevertheless, there is a commonality between the two that arises when the human mind is confronted with new ideas that have some things in common with previous experience and others that have new aspects that need to be addressed.

Taxonomies used for assessment of progress usually specify only positive objectives in the cognitive and affective domains and do not reflect the full range of positive and negative possibilities in the classroom. While some individuals make coherent sense of mathematics and experience the joy of conquering new and more powerful ideas, some learn to cope by working hard to conquer the techniques that they are required to learn and others develop the anxiety of not being able to make sense of the ideas.

It is noteworthy that both the *Common Core State Standards Initiative* in the USA and the *National Curriculum* in England are phrased entirely in positive terms and that in both cases there has been a perceived failure to improve comparative performances in international assessments over the last decade.

The question arises as to whether these two features are related. The focus on attaining successive curriculum targets may have a positive effect on students who are motivated to succeed. In my own case I regarded exams as a positive encouragement to reflect on the course and make sense of the ideas and I consider this as a major reason for my own long-term development. But if a student cannot make sense of a topic in mathematics, this may result in the goal of learning *what* to do to pass the test without understanding *why*.

Van Hiele used the term 'level reduction' for the act of reducing the level of operations from a meaningful conceptual level to a routine procedural level. While this may work for a resilient student willing to work hard to pass an examination, level reduction at one stage may result in a less flexible and more fragmented knowledge structure that makes it more problematic to make sense of new contexts, reinforcing the need for further level reduction. This happens not only to learners but also to teachers who have succeeded in the practical goal of learning procedures to pass examinations and pass on these beliefs to later generations. Focusing on objectives that formulate positive goals may help some individuals to progress, but may also be an impediment for others.

This implies the need to build a joined-up framework for the learning of mathematics, bringing together the cognitive and affective, involving not only the positive aspects that dominate taxonomies used for assessment, but also the full range of emotional reactions that occur in the wider population.

Different communities of practice in mathematics and in psychology have their part to play in the evolution of better ways of teaching and learning mathematical thinking. In recent years, the focus on content in the new math and the focus on the constructive development have each led to a need for change. The current focus on specifying and assessing standards is now under scrutiny.

It is my belief that the next stage of evolution in the teaching and learning of flexible mathematical thinking requires a new blend of summative assessment to focus on the chosen requirements for society, and diagnostic assessment of progress in the cognitive and affective development of individuals to guide them towards effective ways of thinking mathematically. The history of *Enlightening Symbols* and their hidden powers reveals the strengths and cognitive problems of new ideas. Now we need to evolve new methods of teaching, learning and assessing to make sense of mathematics and to evolve mathematical thinking into the future.

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